NCG and RC Brackets

d'après Cohen, Cohen-Manin-Zagier, Eholzer
Connes-Moscovici, Bieliavsky-Rochberg-Tang-Yao

Yi-Jun YAO 姚 一隽

复旦大学

华东师范大学, 2012 年 4 月 17 日

广告 I

- 活动名称: 泛函与拓扑 (Topology and Functional Analysis) 会议
- 活动地点: 复旦大学
- 活动时间: 2012 年 5 月 21-25 日
- 组织委员会: 陈晓漫、龚贵华、郭坤宇、加藤毅、郁国樑、A.Zuk
- 报告人包括: 林华新, N. Ozawa, Goulnara Arzhantseva, Erik van Erp....

- 活动名称: 非交换几何暑期学校
- 活动地点: 复旦大学
- 活动时间: 2012 年 5 月 28 日 -6 月 1 日
- 组织委员会: 陈晓漫、郁国樑
- 主讲人: G.Kasparov, Goulnara Arzhantseva, Erik van Erp, 唐翔, Rufus Willett

- 活动名称: 中法非交换几何暑期学校及 Workshop
- 活动地点: 复旦大学/东华大学/....
- 活动时间: 2012 年 7 月 9 日 -7 月 27 日
- 组织委员会: 陈晓漫、Alain Connes、龚贵华、Georges Skandalis、郁国樑、张伟平
- 主讲人: 法国: Emmanuel Germain, Michel Hilsum, Hervé Oyono-Oyono, Georges Skandalis, Yves Cornulier
  美国: Nigel Higson
  中国: 李寒峰、王勤、郁国樑、张伟平
- 讲课主题: 1. 群胚 C* 代数/非交换指标定理; 2. 离散群和非交换几何

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- $\mathcal{M}(\Gamma) := \sum_k \mathcal{M}_{2k}(\Gamma)$.

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- $[f, g]_n \in \mathcal{M}_{2k+2l+2n}(\Gamma)$.

- (Ramanujan) $X : \mathcal{M}_{2k}(\Gamma) \to \mathcal{M}_{2k+2}(\Gamma)$:

$$\begin{aligned} Xf &= \frac{1}{2\pi i}\frac{df}{dz} - \frac{1}{2\pi i}\frac{\partial}{\partial z}(\log \eta^4)\cdot kf, \\ \eta(z) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \quad q = e^{2\pi i z} \end{aligned}$$

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- Two series of elements by induction: $\Phi \in \mathcal{M}_4(\Gamma)$, $f_0 = f, g_0 = g$,

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- Zagier:

$$\sum_{r=0}^{n}(-1)^r\binom{n+2k-1}{n-r}\binom{n+2l-1}{r}f_r g_{n-r} = [f,g]_n.$$

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Question : Find $c_n(k,l)$ s.t.: $f \in \mathcal{M}_{2k}$, $g \in \mathcal{M}_{2l}$,

$$\mu(f,g) := \sum_{n=0}^{\infty} c_n(k,l)[f,g]_n \hbar^n$$

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Cohen-Manin-Zagier:

$$c_n(k,l) = t_n(k,l) := \frac{1}{\binom{-2l}{n}} \sum_{r+s=n} \frac{\binom{-k}{r}\binom{-k-1}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+l}{s}\binom{n+k+l-1}{s}}{\binom{2n+2k+2l-2}{s}}.$$

Eholzer(conjecture): $c_n(k,l) = 1$ give a solution to the above question.

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- The coefficients are conjectured to be equal to(in order to $\Rightarrow$ Eholzer)

$$t_n^{\kappa}(k,l) = \left(-\frac{1}{4}\right)^n \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-l-\frac{1}{2}}{j}\binom{n+k+l-\frac{3}{2}}{j}}.$$

Hopf algebra for transverse geometry of foliations

In their study of index theory for operators elliptic transverse to a foliation, Connes and Moscovic discovered a sequence of Hopf algebras, $\mathcal{H}_n$, which governs the transversal geometry of a foliation with codim=n.

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Example: Kronecker foliation.

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- Γ-invariant volume form on F^+X : $\omega = \dfrac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$.

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Foliation algebra $\mathcal{A}_\Gamma$: the algebra generated by M_f and U_ϕ, with the relations $U_\phi M_f = M_{\phi^*(f)} U_\phi$. $\mathcal{A}_\Gamma = C_c^\infty(F^+X) \rtimes \Gamma$.

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It is easy to check that Y is invariant under the Γ action, but X is not.

$$U_\phi X U_\phi^{-1} = X - y\frac{{\phi^{-1}}''(x)}{{\phi^{-1}}'(x)}Y.$$

Higher operations

- Linear operator δ_1 on $\mathcal{A}_\Gamma$:

$$\delta_1(fU_\phi) = \mu_{\phi^{-1}} fU_\phi,$$

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- Introduce the sequence of operators δ_n acting on $\mathcal{A}_\Gamma$:

$$\delta_n(fU_\phi) = X^{n-1}(\mu_{\phi^{-1}})fU_\phi.$$

$\mathcal{H}_1$: universal envelopping algebra of the Lie algebra H_1 generated by $X, Y, \delta_n,\ n \in \mathbb{N}$

$$\begin{array}{rcl} [Y, X] = X & , & [Y, \delta_n] = n\,\delta_n\,, \\ [X, \delta_n] = \delta_{n+1} & , & [\delta_k, \delta_\ell] = 0\,, \quad n, k, \ell \geq 1\,. \end{array}$$

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- coproduct : $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$

$$\Delta\, Y = Y \otimes 1 + 1 \otimes Y , \quad \begin{aligned} \Delta\, X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\ \Delta\, \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1 \end{aligned}$$

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- antipod:

$$S(Y) = -Y\,,\ S(X) = -X + \delta_1 Y\,,\ S(\delta_1) = -\delta_1$$

Modular Hecke Algebra

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- *has finite support*, which satisfies *the covariance condition*:

$$F_{\alpha\gamma}(z) = F_\alpha|\gamma(z) = F_\alpha(\gamma.z), \quad \forall \alpha \in GL_2^+(\mathbb{Q}), \gamma \in \Gamma, z \in \mathbb{H}.$$

Modular Hecke Algebra

$\mathcal{A}(\Gamma)$: associative algebra for the product

$$(F^1 * F^2)_\alpha := \sum_{\beta \in \Gamma \backslash GL_2^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}} \Big| \beta.$$

Action of the Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$

- $Xf = \frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{1}{2\pi i}\frac{\partial}{\partial z}(\log \eta^4) \cdot kf \,, \; Y(f) = k \cdot f.$

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- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{Q})$,

$$\mu_\gamma(z) = \frac{1}{2\,\pi^2}\left(G_2^*|\gamma(z) - G_2^*(z) + \frac{2\pi i\, c}{cz+d}\right)$$

$$G_2^*(z) = 2\zeta(z) + 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} = \frac{\pi^2}{3} - 8\pi^2\sum_{m,n\geq 1} me^{2\pi imnz}.$$

- Notice here $\mu_\alpha \equiv 0$ if $\alpha \in SL_2(\mathbb{Z})$.

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A_{n+1} &:= S(X)\, A_n - n\, \Omega^o \left(Y - \frac{n-1}{2} \right) A_{n-1}\,, \\
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$$RC_n(a,b) := \sum_{k=0}^{n} \frac{A_k}{k!}\,(2Y+k)_{n-k}(a)\,\frac{B_{n-k}}{(n-k)!}\,(2Y+n-k)_k(b).$$

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- **Theorem**(Connes-Moscovici) The functor $RC_* := \sum RC_n$ applied to any algebra $\mathcal{A}$ endowed with a projective structure yields a family of formal associative deformations of $\mathcal{A}$, whose products are given by

$$f \star g = \sum RC_n(f, g)\hbar^n.$$

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- The Moyal Product: for $a, b \in W_x$,

$$\begin{aligned} a \circ b &= \exp\left(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(y, \hbar) b(z, \hbar)|_{z=y} \\ &= \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!}\omega^{i_1 j_1} \cdots \omega^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}}. \end{aligned}$$

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- **Corollary** $W_D \leftrightarrow C^\infty(M)[[\hbar]]$. We can then define on $C^\infty(M)[[\hbar]]$ an associative product

$$a \star b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)).$$

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Question: Explicit formula?

Construction of Discrete Series

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- representation space $\xrightarrow{(\sigma_{2(k+n)})^{-1},\, n \geq 0} \subset C^{\infty}(\mathbb{H})$.

Theorem. *Let* $f \in \mathcal{M}_{2k}, g \in \mathcal{M}_{2l}$ *be two modular forms. Let* $\pi_f \cong \pi_{\deg f}, \pi_g \cong \pi_{\deg g}$ *be the corresponding discrete series representations of the Lie group* $SL_2(\mathbb{R})$*.*

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The Rankin-Cohen bracket $[f, g]_n$ gives (up to scale) the vectors of minimal K-weight in the representation space of the component $\pi_{\deg f + \deg g + 2n}$.

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- Remarque 2.1.4. L'espace $F(G, GL(2,\mathbb{Z}))$ ci-dessus est stable par produit. D'autre part, $D_{k-1} \otimes D_{l-1}$ contient les $D_{k+l+2m-1}(m \geq 0)$. Pour$m = 0$, ceci correspond au fait que le produit fg d'une forme modulaire holomorphe de poids k par une de poids l, en est une de poids $k+l$. Pour $m = 1$, en coordonnées (1.1.5.2), on trouve que $l\frac{\partial f}{\partial z}.g - kf.\frac{\partial g}{\partial z}$ est modulaire holomorphe de poids $k+l+2$, et ainsi de suite. De même dans le cadre adélique.

Explicit Formulae

$$\mathcal{M}(\Gamma) \subset \tilde{\mathcal{M}}(\Gamma) := \left\{ f \,,\, \exists k \in \mathbb{N} \,,\, f(z) = \left(f\Big|_{2k} \gamma \right)(z) \right\}.$$

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$$RC_{(X,Y,\delta_k)} = RC_{red,(\tilde{X},Y)}.$$

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$\star : \tilde{\mathcal{M}}(\Gamma)^{\otimes}[[\hbar]] \times \tilde{\mathcal{M}}(\Gamma)^{\otimes}[[\hbar]] \to \tilde{\mathcal{M}}(\Gamma)^{\otimes}[[\hbar]]$: bilinear extension +

$$f \star g = \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} \left(\sum_{r=0}^{n} (-1)^r \tilde{X}^r \binom{\deg f + n - 1}{n - r} f \otimes \tilde{X}^{n-r} \binom{\deg g + n - 1}{r} g \right) \hbar^n,$$

where $f, g \in \tilde{\mathcal{M}}(\Gamma)^{\otimes}$.

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- $\to$ Strong Associativity.

The restriction of $\star$ to $\tilde{\mathcal{M}}(\Gamma) \subset \tilde{\mathcal{M}}(\Gamma)^{\otimes}$ composed with M.
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- $\to$ Weak Associativity.

(Yao)

$$\text{Strong Associativity} \Longleftrightarrow \text{Weak Associativity.}$$

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- Strong Associativity
- $$\Updownarrow$$

for $p = 0, 1, \ldots, n$:

$$\sum_{r=0} \binom{n-r}{p} \frac{A_{n-r}(2k+2l+2r, 2m)A_r(2k, 2l)}{(2k+2l+2r)_{n-p-r}(2m)_p(2k)_r}$$

$$= \sum_{s=0} \binom{n-s}{n-p} \frac{A_{n-s}(2k, 2l+2m+2s)A_s(2l, 2m)}{(2k)_{n-p}(2l+2m+2s)_{p-s}(2m)_s}.$$

Theorem(Yao). *Cohen-Manin-Zagier have found all the associative formal products* $* : \tilde{\mathcal{M}}(\Gamma)[[\hbar]] \times \tilde{\mathcal{M}}(\Gamma)[[\hbar]] \to \tilde{\mathcal{M}}(\Gamma)[[\hbar]]$ *defined by linearity and the formula*

$$f * g \quad = \quad \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n}[f, g]_n \hbar^n, \qquad (1)$$

where $\tilde{\mathcal{M}}$ *is the space of the functions which satisfy the modularity condition, and the notation* $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$. *We assume moreover* $A_0 = 1$ *and* $A_1(x, y) = xy$*,*

Proposition(Yao). *Let* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$ *such that* $\mathcal{M}(\Gamma)$ *admit the unique factorization property (for example* $SL_2(\mathbb{Z})$ *itself), let* $F_1, F_2, G_1, G_2 \in \mathcal{M}(\Gamma)$ *such that*

$$RC(F_1, G_1) = RC(F_2, G_2),$$

as formal series in $\mathcal{M}(\Gamma)[[\hbar]]$*, then there exists a constant* C *such that*

$$F_1 = CF_2, G_2 = CG_1.$$

Ex. we want to show that there does not exist any **positive integer** roots of the multivariable polynomial

$$
\begin{aligned}
& P_3(k,l,m,r,t) \\
= \ & 4l(r+t)(-3k^2r^2 - 2k^3r^2 + 3klr^2 + 2kl^2r^2 - 6kmr^2 - 15k^2mr^2 - 3k^3mr^2 + 3lmr^2 \\
& -9klmr^2 - 6k^2lmr^2 3kl^2mr^2 - 3m^2r^2 - 24km^2r^2 - 15k^2m^2r^2 - 9lm^2r^2 \\
& -24klm^2r^2 - 9l^2m^2r^2 - 11m^3r^2 - 21km^3r^2 - 18lm^3r^2 - 9m^4r^2 + 12k^2rt + 17k^3rt \\
& +3k^4rt + 6klrt + 21k^2lrt + 6k^3lrt + 4kl^2rt + 3k^2l^2rt + 24kmrt + 51k^2mrt + 24k^3mrt \\
& +6lmrt + 42klmrt + 42k^2lmrt + 4l^2mrt + 18kl^2mrt + 12m^2rt + 51km^2rt + 42k^2m^2rt \\
& +21lm^2rt + 42klm^2rt + 3l^2m^2rt + 17m^3rt + 24km^3rt + 6lm^3rt + 3m^4rt - 3k^2t^2 \\
& -11k^3t^2 - 9k^4t^2 + 3klt^2 - 9k^2lt^2 - 18k^3lt^2 + 2kl^2t^2 - 9k^2l^2t^2 - 6kmt^2 - 24k^2mt^2 \\
& -21k^3mt^2 + 3lmt^2 - 9klmt^2 - 24k^2lmt^2 + 2l^2mt^2 - 3kl^2mt^2 - 3m^2t^2 - 15km^2t^2 \\
& -15k^2m^2t^2 - 6klm^2t^2 - 2m^3t^2 - 3km^3t^2 + 2l^2mr^2).
\end{aligned}
$$

where $t = \mu[(k+3m)(k+l+m)+(k+m)], r = \mu[(3k+m)(k+l+m)+(k+m)]$.

$$
\begin{aligned}
& P_3(k,l,m,\mu[(3k+m)(k+l+m)+(k+m)],\mu[(3k+m)(k+l+m)+(k+m)]) \\
= \;& \mu^3(48k^5l+320k^6l+720k^7l+672k^8l+256k^9l+96k^4l^2+960k^5l^2+2976k^6l^2+3552k^7l^2 \\
& +1536k^8l^2+640k^4l^3+3792k^5l^3+6624k^6l^3+3584k^7l^3+1536k^4l^4+5280k^5l^4 \\
& +4096k^6l^4+1536k^4l^5+2304k^5l^5+512k^4l^6+240k^4lm+1920k^5lm+5232k^6lm \\
& +5760k^7lm+2304k^8lm+384k^3l^2m+4800k^4l^2m+18240k^5l^2m+26016k^6l^2m \\
& +12288k^7l^2m+2560k^3l^3m+19152k^4l^3m+40896k^5l^3m+25088k^6l^3m+6144k^3l^4m \\
& +26784k^4l^4m+24576k^5l^4m+6144k^3l^5m+11520k^4l^5m+2048k^3l^6m+480k^3lm^2 \\
& +4800k^4lm^2+16080k^5lm^2+21120k^6lm^2+9216k^7lm^2+576k^2l^2m^2+9600k^3l^2m^2 \\
& +46176k^4l^2m^2+80352k^5l^2m^2+43008k^6l^2m^2+3840k^2l^3m^2+38496k^3l^3m^2 \\
& +103968k^4l^3m^2+75264k^5l^3m^2+9216k^2l^4m^2+53952k^3l^4m^2+61440k^4l^4m^2 \\
& +9216k^2l^5m^2+23040k^3l^5m^2+3072k^2l^6m^2+480k^2lm^3+6400k^3lm^3+27120k^4lm^3 \\
& +43392k^5lm^3+21504k^6lm^3+384kl^2m^3+9600k^2l^2m^3+61824k^3l^2m^3+135840k^4l^2m^3 \\
& +86016k^5l^2m^3+2560kl^3m^3+38496k^2l^3m^3+139392k^3l^3m^3+125440k^4l^3m^3 \\
& +6144kl^4m^3+53952k^2l^4m^3+81920k^3l^4m^3+6144kl^5m^3+23040k^2l^5m^3 \\
& +2048kl^6m^3+240klm^4+4800k^2lm^4+27120k^3lm^4+54720k^4lm^4+256lm^9 \\
& +32256k^5lm^4+96l^2m^4+4800kl^2m^4+46176k^2l^2m^4+135840k^3l^2m^4 \\
& +107520k^4l^2m^4+640l^3m^4+19152kl^3m^4+103968k^2l^3m^4+125440k^3l^3m^4 \\
& +1536l^4m^4+26784kl^4m^4+61440k^2l^4m^4+1536l^5m^4+11520kl^5m^4+512l^6m^4 \\
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& +75264k^2l^3m^5+5280l^4m^5+24576kl^4m^5+2304l^5m^5+320lm^6+5232klm^6 \\
& +21120k^2lm^6+21504k^3lm^6+2976l^2m^6+26016kl^2m^6+43008k^2l^2m^6 \\
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- $SL(2, \mathbb{R})$ action:

$$\pi_{2k}(\gamma)(f)(z) = f(\frac{az+b}{cz+d})(cz+d)^{-\alpha}.$$

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Henkel form: $SL_2(\mathbb{R})$ equivariant bilinear map from $A^{2k}(D) \times A^{2l}(D)$ to $A^{2k+2l+2i}$ that corresponds to the projection from $A^{2k}(D) \otimes A^{2l}(D)$ to $A^{2k+2l+2i}$.

$$\Pi_{2i}(f,g)(z) = \sum_{s=0}^{i} (-1)^{i-s} \begin{pmatrix} i \\ s \end{pmatrix} \frac{1}{(w(f))_s (w(g))_{i-s}} \partial^s f \partial^{i-s} g,$$

where $w(f)$ and $w(g)$ are weights of f and g.

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- $\mathcal{B}_\Gamma = A_{T^{1,0}\Sigma} \rtimes \Gamma$.

Proposition(Rochberg-Tang-Yao). *Connes-Moscovici's Hopf algebra* $\mathcal{H}_1$ *acts naturally on the algebra* $\mathcal{B}_\Gamma$. *The reduced* i*-th Rankin-Cohen bracket on* $\oplus_{n\geq 0} A^{2n}(D)$ *is a Hankel form of weight* $2i$.

- Moyal product : $f, g \in \mathcal{S}(\mathbb{R}^2)$,

$$f * g = \sum_n \frac{\hbar^n}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\partial^n f}{\partial x^i \partial y^{n-i}} \frac{\partial^n g}{\partial x^{n-i} \partial y^i}.$$

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- Weyl product : $f, g \in \mathcal{S}(\mathbb{R}^2)$,

$$(f *^W g)(x, y) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x + u_1, y + u_2) g(x + v_1, y + v_2) \, \mathrm{e}^{2\pi \mathrm{i}(u_1 v_2 - u_2 v_1)} \, \mathrm{d}^2 u \mathrm{d}^2 v.$$

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- Weyl $\xrightarrow{\text{asymptotic expansion}}$ Moyal

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- K-theory of the deformed algebra is the same as that of the original one.

Moyal

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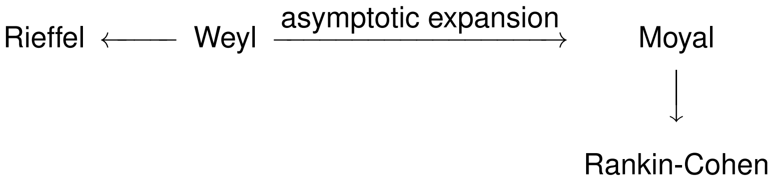
Rieffel
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$$
\begin{array}{ccccc}
\text{Rieffel} & \longleftarrow & \text{Weyl} & \xrightarrow{\text{asymptotic expansion}} & \text{Moyal} \\
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谢谢!